A GENERALIZED THEORY OF DYNAMICAL TRAJECTORIES

BY

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1. **Introduction.** The differential geometry of the dynamical trajectories of positional fields of force has been developed in the Princeton Colloquium(1). The fields of force, considered heretofore, depend only upon the position of the point. We shall call a field of force *generalized* if it depends not only upon the position of the point but also upon the direction through the point. In this paper, it is our purpose to begin the development of the geometry of the dynamical trajectories of such generalized fields of force in the plane. Positional fields may be described as isotropic and generalized fields as *anisotropic*.

In a generalized field of force, there are ∞^3 dynamical trajectories. These are represented by a differential equation of the form

$$\text{(G)} \qquad y''' = G(x, y, y')y'' + H(x, y, y')y''^2.$$

Conversely, any system of curves of the type (G) may represent the dynamical trajectories of $\infty^{f(2)}$ generalized fields of force(2). We therefore find it appropriate to term any system of curves of the type (G) as *generalized dynamical trajectories*.

Of course, not every such system of curves represents the trajectories of a positional field of force. Kasner proved that the family of ordinary dynamical trajectories is characterized by a set of *five* independent geometric properties. Of these, the Property I is equivalent analytically to stating that a system of ordinary dynamical trajectories must be represented by a differential equation of the type (G). The other four geometric properties specialize the functions G and H.

The Property I is described as follows. For each of the ∞^1 curves of the ∞^3 trajectories which pass through a given lineal element E, construct the osculating parabola at E. The Property I is given by any one of the following three equivalent statements(3).

Presented to the Society, October 31, 1942; received by the editors August 28, 1942.

(1) Kasner, *Differential-geometric aspects of dynamics*, Amer. Math. Soc. Colloquium Publications, vol. 3, 1913, 1934 (referred to as Princeton Colloquium). Also see a series of papers in Trans. Amer. Math. Soc. vols. 7–11 (1906–1910).

(2) The symbol $\infty^{f(2)}$ means that the force vector contains an arbitrary function of two variables. See Kasner, *A notation for infinite manifolds*, Amer. Math. Monthly vol. 49 (1942) pp. 243–244.

(3) Recently Terracini has given an alternate projective characterization of Property I by conic sections. See *Sobre la ecuacion diferencial* $y'''=G(x, y, y')y''+H(x, y, y')y''^2$, Revista de Matematicas vol. 2 (1941) pp. 245–329.

(I_a) The locus of the foci is a circle through the point of E.

(I_b) The directrices form a pencil of lines.

(I_c) The envelope of the ∞^1 parabolas is a straight line.

Next we shall compare the ∞^1 lines of force and the ∞^2 rest trajectories. Let a particle which is initially at rest start from a point O. The rest trajectory will then be tangent to the line of force at O. We study the ratio ρ of the departure of the rest trajectory to that of the line of force at O. In the positional case, Kasner showed that $\rho=1/3$, or, if the order of contact of each of the curves with their common tangent line is n, $\rho=1/(2n+1)$. We obtain a converse theorem determining all generalized fields of force for which this ratio assumes the above values. (See formulas (35) and (45).)

Finally we shall define for our generalized fields of force other systems of curves, namely, velocity systems, systems S_k, and pressure curves. Each of these systems is of the type (G) and hence possesses the Property I. While the systems S_k and the pressure curves may be represented by a differential equation of the type (G), it is found that velocity systems must be curvature trajectories(4).

2. **The differential equation of the generalized dynamical trajectories.** Let ϕ and ψ be the rectangular components parallel to the x- and y-axes of the generalized force vector acting at the lineal element (x, y, y')(5). We assume that our force vector (ϕ, ψ) does not identically agree with the direction of the corresponding lineal element E. For, if $\psi - y'\phi \equiv 0$, the trajectories are straight lines, and are, therefore, of no interest. The equations of motion of a particle of unit mass are

$$(1) \qquad d^2x/dt^2 = \phi(x, y, y'), \qquad d^2y/dt^2 = \psi(x, y, y').$$

The particle may be started from any position (x_0, y_0) with any velocity $(dx_0/dt_0, dy_0/dt_0)$. A definite trajectory is described. Since the same curve may be obtained by starting from any one of its ∞^1 points, the total number of trajectories, for all initial conditions, is ∞^3.

We now proceed to eliminate the time t from the equations (1). (For the positional case, see the Princeton Colloquium, pp. 7–9.) The elimination can be performed directly from (1), but we prefer to do it in the following manner. Let N and T be the normal and tangential components along a trajectory. Then

$$(2) \qquad \begin{aligned} N &= (\psi - y'\phi)/(1 + y'^2)^{1/2}, \qquad & T &= (\phi + y'\psi)/(1 + y'^2)^{1/2}; \\ \phi &= (T - y'N)/(1 + y'^2)^{1/2}, \qquad & \psi &= (N + y'T)/(1 + y'^2)^{1/2}. \end{aligned}$$

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(4) Kasner, *Dynamical trajectories and curvature trajectories*, Bull. Amer. Math. Soc. vol. 40 (1934) pp. 449–455.

(5) The functions ϕ and ψ could have been assumed to involve x and y and also to be homogeneous of degree zero in dx/dt and dy/dt. Then since during motion dx/dt and dy/dt are not both zero, the functions ϕ and ψ are seen to depend on (x, y, y') only.

By these equations, it is observed that either ϕ and ψ, or $N\neq 0$ and T, are two arbitrary functions of (x, y, y').

If v denotes the speed of the particle and r the radius of curvature, we find

$$v^2 = rN, \qquad vv_s = T. \tag{3}$$

Eliminating the speed v from these equations, we find the intrinsic differential equation of the generalized dynamical trajectories to be

$$d(rN)/ds = 2T. \tag{4}$$

Upon putting this equation into the cartesian form, we find the following result.

THEOREM 1. *The* ∞^3 *dynamical trajectories of a generalized field of force are given by the differential equation*

$$(\psi - y'\phi)y''' = [\psi_x + y'(\psi_y - \phi_x) - y'^2\phi_y]y'' + [\psi_{y'} - y'\phi_{y'} - 3\phi]y''^2. \tag{5}$$

This is of type (G).

It is noted that this represents the ordinary dynamical type if ϕ and ψ are independent of y'.

3. **The type (G) always represents the trajectories of a generalized field of force.** In order to simplify our discussion for generalized fields of force, it is convenient to introduce the functions α and β defined as follows:

$$\begin{aligned} e^\alpha &= (1 + y'^2)^{3/2}N, & \beta &= -2T/(1 + y'^2)N; \\ N &= e^\alpha/(1 + y'^2)^{3/2}, & T &= -e^\alpha\beta/2(1 + y'^2)^{1/2}. \end{aligned} \tag{6}$$

We shall also write the equations connecting (ϕ, ψ) and (α, β) as follows:

$$\begin{aligned} e^\alpha &= (1 + y'^2)(\psi - y'\phi), & \beta &= -2(\phi + y'\psi), \\ \phi &= -\frac{[2y'e^\alpha + (1 + y'^2)\beta]}{2(1 + y'^2)^2}, & \psi &= \frac{[2e^\alpha - y'(1 + y'^2)\beta]}{2(1 + y'^2)^2}. \end{aligned} \tag{7}$$

Now placing the last two of equations (6) into the differential equation (5), we see that the differential equation of the generalized dynamical trajectories is

$$y''' = (\alpha_x + y'\alpha_y)y'' + (\alpha_{y'} + \beta)y''^2. \tag{8}$$

THEOREM 2. *Any system of curves of the type* (G) *represents the dynamical trajectories of* $\infty^{f(2)}$ *generalized fields of force.*

We proceed to the proof of our theorem. Let a system of curves of the type (G) be given. If there exists a generalized field of force of which the given curves are the trajectories, the functions α and β must be determined so that

$$\alpha_x + y'\alpha_y = G, \qquad \alpha_{y'} + \beta = H. \tag{9}$$

By the theory of differential equations, we know that the first equation determines α as an expression of the form

$$\alpha = \alpha_1(x, y, y') + \alpha_2(y - xy', y'), \tag{10}$$

where α_1 is a definite function of three variables and α_2 is an arbitrary function of two variables. The second equation then completely determines β. This completes the proof of our Theorem 2.

For example, all generalized fields of force whose dynamical trajectories are the vertical parabolas $y = ax^2 + bx + c$, are determined by

$$\alpha = F(y - xy', y'), \qquad \beta = xF_{y-xy'} - F_{y'}. \tag{11}$$

4. **Fields of force depending only upon differential elements of order** n **for** $n \geqq 2$. The argument presented in the preceding section may be generalized to prove the following result.

THEOREM 3. *Any system of ∞^{n+1} curves for $n \geqq 2$ may represent the dynamical trajectories of $\infty^{f(n+2)}$ more extensive fields of force depending only upon differential elements of order n.*

The equations of motion of a particle of unit mass for a field of force which depends only upon the position of a differential element of order n, are

$$d^2x/dt^2 = \phi(x, y, y', \cdots, y^{(n)}), \qquad d^2y/dt^2 = \psi(x, y, y', \cdots, y^{(n)}). \tag{12}$$

Now if N and T represent the normal and tangential components of the force vector along any trajectory, it is found by equations analogous to (2) that N and T may be considered to be any two arbitrary functions of $(x, y, y', \cdots, y^{(n)})$. Then since $n \geqq 2$, the functions N^* and T^* defined by

$$N^* = rN, \qquad T^* = 2T(1 + y'^2)^{1/2}, \tag{13}$$

are also arbitrary functions of $(x, y, y', \cdots, y^{(n)})$.

The intrinsic differential equation of the trajectories of such an extensive field of force is given by an equation analogous to (4). Therefore replacing rN and $2T$ by the functions N^* and $T^*(1+y'^2)^{-1/2}$ and performing the differentiation, the differential equation of the dynamical trajectories is

$$y^{(n+1)} = (1/N^*_{y^{(n)}})\left[T^* - (N^*_x + y'N^*_y + y''N^*_{y'} + \cdots + y^{(n)}N^*_{y^{(n-1)}})\right]. \tag{14}$$

Therefore there are ∞^{n+1} dynamical trajectories in an extensive field of force depending only upon the position of a differential element of order $n \geqq 2$.

The right-hand side of (14) may represent any arbitrary function of $(x, y, y', \cdots, y^{(n)})$. Also the choice of T^* is uniquely determined as soon as

N^* is given. Therefore there are $\infty^{f(n+2)}$ extensive fields of force whose trajectories are a given family of ∞^{n+1} curves.

5. **The geometric Property** I. Before discussing the Property I, it is advisable to collect here the various formulas for the osculating parabola of a curve(6). The equation in running coordinates (X, Y) of the osculating parabola at any differential element of the third order (x, y, y', y'', y''') of the curve $y=y(x)$ is

$$\begin{aligned}[(y'y''' - 3y''^2)(X - x) - y'''(Y - y)]^2 \\ + 18y''^3[y'(X - x) - (Y - y)] = 0.\end{aligned} \tag{15}$$

The focus is

$$X + iY = x + iy + \frac{3y''(1 + iy')^2}{2[y'''(1 + iy') - 3iy''^2]} \cdot \tag{16}$$

The equation of the directrix is

$$2y'''(X - x) + 2(y'y''' - 3y''^2)(Y - y) = 3y''(1 + y'^2). \tag{17}$$

Consider now any system of ∞^3 curves. For each of the ∞^1 curves of this system which pass through a given lineal element E, construct the osculating parabola at E. This gives rise to ∞^1 parabolas all containing the lineal element E.

THEOREM 4. *A system of ∞^3 curves is a generalized dynamical system if and only if the locus of the foci of the ∞^1 osculating parabolas, constructed at any lineal element E, is a circle passing through the point of E.*

The equation of this focal circle is

$$\begin{aligned}2G[(X - x)^2 + (Y - y)^2] - [y'(1 + y'^2)H + 3(1 - y'^2)](X - x) \\ + [(1 + y'^2)H - 6y'](Y - y) = 0.\end{aligned} \tag{18}$$

THEOREM 5. *A system of ∞^3 curves is a generalized dynamical system if and only if the directrices of the ∞^1 osculating parabolas, constructed at any lineal element E, form a pencil of lines.*

The vertex D of this pencil of lines is

$$D\colon\ X + iY = x + iy + \frac{(1 + y'^2)}{2G}[-(y'H - 3) + iH]. \tag{19}$$

Consider now any system of ∞^3 curves. At a fixed lineal element E, y''' is a function of y'' only. Now let ω represent the expression y'''/y''^2 so that at E,

(6) For an analogous discussion in minimal coordinates, see Kasner and DeCicco, *Families of curves conformally equivalent to circles*, Trans. Amer. Math. Soc. vol. 49 (1941) pp. 378–391.

ω is a function of y'' only. By (15) the ∞^1 osculating parabolas at E are represented by the equation

$$(20)\qquad y''[(y'\omega - 3)(X - x) - \omega(Y - y)]^2 + 18[y'(X - x) - (Y - y)] = 0.$$

Now if $\omega_{y''} = 0$, our family of parabolas is linear. The foci will all lie on the straight line into which the focal circle (18) degenerates by letting G approach zero. The directrices will all be parallel, their directions being perpendicular to the direction so that the line of E bisects the angle at the point O of E formed by this direction and the focal straight line. The complete envelope of this linear family of parabolas consists of the line of E and the line at infinity.

Any system of ∞^3 curves such that the ∞^1 osculating parabolas, constructed at E, form a linear family must be of the special subclass of type (G)

$$(21)\qquad y''' = H(x, y, y')y''^2.$$

Let now $\omega_{y''} \neq 0$. Thence our family of ∞^1 parabolas (20) possesses an envelope (in addition to the common tangent line of the element E). This envelope is given parametrically by the equations

$$(22)\qquad X - x = -\frac{3(\omega + 2y''\omega_{y''})}{2y''^3\omega^2_{y''}}, \quad (Y - y) - y'(X - x) = \frac{9}{2y''^3\omega^2_{y''}}.$$

Now differentiating these partially with respect to y'', we find

$$(23)\qquad \begin{aligned} X_{y''} &= \frac{3}{2y''^4\omega^3_{y''}}(3\omega_{y''} + 2y''\omega_{y''y''})(\omega + y''\omega_{y''}), \\ Y_{y''} - y'X_{y''} &= -\frac{9}{2y''^4\omega^3_{y''}}(3\omega_{y''} + 2y''\omega_{y''y''}). \end{aligned}$$

Our envelope is obviously a point if and only if $3\omega_{y''} + 2y''\omega_{y''y''} = 0$. Therefore the family of ∞^1 osculating parabolas, constructed at the element E, pass through a fixed point if and only if the differential equation of the system of ∞^3 curves is of the form

$$(24)\qquad y''' = \mu(x, y, y')y''^{3/2} + \nu(x, y, y')y''^2.$$

Now we exclude this situation. Then the slope dY/dX of our envelope (21) is given by the equation

$$(25)\qquad dY/dX - y' = -3/(\omega + y''\omega_{y''}).$$

This formula demonstrates that our envelope is a straight line if and only if $y''\omega = G + Hy''$. Therefore we may state the following result.

THEOREM 6. *A system of* ∞^3 *curves is a generalized dynamical system if and* *only if the envelope of the* ∞^1 *osculating parabolas, constructed at any lineal element* E, *is a straight line* (*not passing through the point of* E).

This common tangent line γ of the ∞^1 osculating parabolas is given by the equation

$$2G(y'H - 3)(X - x) - 2GH(Y - y) + 9 = 0. \tag{26}$$

Thus our ∞^1 parabolas are tangent to the line γ, the line of the element E at the point of E, and the line at infinity.

Our line γ intersects the line of the lineal element E in the point

$$P: \quad X + iY = x + iy + (3/2G)(1 + iy'). \tag{27}$$

Moreover we find that our line γ is tangent to the focal circle at this point P. The line DP connecting the vertex D of the directrices and this point P, is perpendicular to the line of E.

We remark that the one-parameter set of parabolas for a given element E, is in general quadratic.

6. **The angular rate λ and the terminal curve** C. For the further development of our theory, it is found essential in a given generalized field of force to associate a number λ to any lineal element E. As the element E rotates about its point O, the corresponding force vector F also rotates about O. We define the *angular rate* λ as the instantaneous rate of change of the inclination of F with respect to the inclination of E. It is given by the formula

$$\lambda = \frac{1 + y'^2}{\phi^2 + \psi^2}(\phi\psi_{y'} - \psi\phi_{y'}). \tag{28}$$

Also as E rotates about its point O, the end point of the corresponding force vector F describes a curve C. We shall call this the *terminal curve* C corresponding to the point O. The terminal curve C degenerates into a fixed point if and only if our field of force is ordinary positional.

Obviously for an ordinary positional field of force the angular rate $\lambda \equiv 0$. However there are also certain generalized fields of force for which the angular rate $\lambda \equiv 0$. These fields are such that the terminal curve corresponding to any point O is a straight line through O.

7. **The lines of force and the rest trajectories.** In this section, we wish to define two special important families of curves in a given generalized field of force, namely, the lines of force and the rest trajectories. Let us first consider the lines of force. In general, at a given point O, there is one lineal element E_0 such that the direction of the force vector F_0 is identical with that of E_0. Let us also suppose that the angular rate λ at such an element E_0 is not unity. If in a certain region of the plane, there are ∞^2 such lineal elements E_0, we define the integral curves of this set of ∞^2 elements E_0 as the *lines of force* of our generalized field of force.

The lines of force are given by the differential equation of the first order

$$\psi(x, y, y') - y'\phi(x, y, y') = 0. \tag{29}$$

The condition that the angular rate $\lambda \neq 1$ is equivalent to stating that there exists a function $y' = f(x, y)$ which satisfies this equation identically. Thus in general, there are ∞^1 lines of force.

A *rest trajectory* is defined as the path described in a generalized field of force by a particle which starts at rest from a given point O. From the equations

$$dy/dt = y'\,dx/dt, \qquad d^2y/dt^2 = y''(dx/dt)^2 + y'\,d^2x/dt^2, \tag{30}$$

together with the initial conditions $dx_0/dt_0 = 0$, $dy_0/dt_0 = 0$, it is immediately deduced that a rest trajectory is always initially tangent to the line of force through the given point O. Since on any rest trajectory, there is, in general, only one such point O where the particle is at rest, it is deduced that there are ∞^2 rest trajectories in a given generalized field of force(7).

8. **The generalization of the theorem concerning one-third of the curvatures.**(8) We shall now study in a given generalized field of force the ratio ρ of the departure of the rest trajectory to that of the line of force at a given point O. For the ordinary case Kasner showed that this ratio $\rho = 1/3$. We shall find, however, a more general formula for ρ in terms of the angular rate λ.

At the point O, let E_0 be the lineal element of the line of force. Then the rest trajectory described by a particle initially at rest at O will be tangent to the line of E_0.

Now we shall assume that the curvature of the rest trajectory is not zero at the point O. Let us calculate the value of the second derivative y'' of the rest trajectory at O. Since our rest trajectory is tangent to the line of force at O, the first derivative y' must satisfy the equation (29). Therefore from this and the equation (5), it results that the second derivative y'' of the rest trajectory at O is given by the formula

$$[\psi_x + y'(\psi_y - \phi_x) - y'^2\phi_y] + [\psi_{y'} - y'\phi_{y'} - 3\phi]y'' = 0. \tag{31}$$

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(7) Of course, we are assuming that in the region of the *opulence* (the totality of ∞^3 lineal elements of the plane) where the generalized field of force is defined, there are points O (even in the imaginary domain) through which there pass such special lineal elements E_0 (one or more). Otherwise there are no theories of lines of force and rest trajectories.

(8) For a generalization to acceleration fields of higher order, see Kasner and Mittleman, *A general theorem on the initial curvature of dynamical trajectories*, Proc. Nat. Acad, Sci. U.S.A. vol. 28 (1942) pp. 48–52; see also *Extended theorems in dynamics*, Science vol. 95 (1942) pp. 249–250.

Also see Fialkow, *Initial motion in fields of force*, Trans. Amer. Math. Soc. vol. 40 (1936) pp. 495–501; and Kasner and Fialkow, Trans. Amer. Math. Soc. vol. 41 (1937).

The second derivative Y'' of the line of force is calculated by taking the total derivative of the equation (29). Therefore this is given by

$$[\psi_x + y'(\psi_y - \phi_x) - y'^2\phi_y] + [\psi_{y'} - y'\phi_{y'} - \phi]Y'' = 0. \tag{32}$$

Subtracting these two equations, we find

$$[\psi_{y'} - y'\phi_{y'} - 3\phi]y'' = [\psi_{y'} - y'\phi_{y'} - \phi]Y''. \tag{33}$$

Since the magnitude of the force by assumption is not zero and since at E_0, $\psi = y'\phi$, we see that $\phi \neq 0$. Upon dividing this equation by ϕ and making use of the equation (28) defining our angular rate λ, we see that the preceding equation may be written in the form

$$(3 - \lambda)y'' = (1 - \lambda)Y''. \tag{34}$$

The angular rate $\lambda \neq 1$. For, otherwise, the line of force could not exist. Also if the angular rate $\lambda \neq 3$, the curvature of the line of force is not zero. The order of contact with the common tangent line of the line of force will be two or more if $\lambda = 3$. Thus we may state the following result.

THEOREM 7. *The ratio ρ of the curvatures at E_0 of the rest trajectory and the line of force is given by the formula*

$$\rho = (1 - \lambda)/(3 - \lambda) \cdot \tag{35}$$

For an ordinary field of force, the angular rate $\lambda = 0$ so that $\rho = 1/3$. There are generalized fields of force for which the ratio $\rho = 1/3$. We shall determine these in a later section.

9. The generalization of the theorem concerning $1/(2n+1)$ of the rates of departure. In this section, we shall generalize our Theorem 7 to the case where the rest trajectory has contact of order $n \geqq 2$ with the line of the lineal element E_0 at the point O of E_0. For this purpose, it is found advisable to take the element E_0 in the normal form, that is, its point O at the origin and its direction that of the x-axis. Therefore our rest trajectory may be written in the form

$$y = \frac{c_{n+1}}{(n+1)!}x^{n+1} + \frac{c_{n+2}}{(n+2)!}x^{n+2} + \cdots . \tag{36}$$

The line of force has at least first order contact with the line of E_0. Let m denote its order of contact with this line. Therefore our line of force may be written as

$$y = \frac{C_{m+1}}{(m+1)!}x^{m+1} + \frac{C_{m+2}}{(m+2)!}x^{m+2} + \cdots . \tag{37}$$

The equation (36) must satisfy the differential equation (5). In order to perform this substitution, we need to know the expansion of the left-hand side of (29) when (36) is placed into it. For that purpose, it is necessary to know the total derivative of order p of the left-hand side of (29).

To solve this problem, it is convenient to introduce the following notation. If χ is any function of (x, y, y'), then

$$\chi^{(1)} = \left(\frac{\partial}{\partial x} + y'\frac{\partial}{\partial y}\right)\chi, \cdots, \chi^{(p)} = \left(\frac{\partial}{\partial x} + y'\frac{\partial}{\partial y}\right)^p \chi. \tag{38}$$

With this notation, we see that the first p derivatives of the left-hand side of (29) are

$$\begin{aligned}
&\frac{d}{dx}(\psi - y'\phi) = \psi^{(1)} - y'\phi^{(1)} + y''(\psi_{y'} - y'\phi_{y'} - \phi),\\
&\frac{d^2}{dx^2}(\psi - y'\phi) = \psi^{(2)} - y'\phi^{(2)} + y''a_{2,2} + y'''(\psi_{y'} - y'\phi_{y'} - \phi),\\
&\frac{d^3}{dx^3}(\psi - y'\phi) = \psi^{(3)} - y'\phi^{(3)} + y''a_{3,3} + y'''a_{3,2} + y^{\mathrm{IV}}(\psi_{y'} - y'\phi_{y'} - \phi),\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&\frac{d^p}{dx^p}(\psi - y'\phi) = \psi^{(p)} - y'\phi^{(p)} + y''a_{p,p} + y'''a_{p,p-1} + y^{\mathrm{IV}}a_{p,p-2}\\
&\qquad + \cdots + y^{(p)}a_{p,2} + y^{(p+1)}(\psi_{y'} - y'\phi_{y'} - \phi),
\end{aligned} \tag{39}$$

where $a_{p,q} = f_p(x, y, y', y'', \cdots, y^{(q)})$ only. This last formula may easily be proved by induction.

If the line of force has contact of order m with the line of E_0, not necessarily in the normal position, then from the preceding equations, it follows that

$$\begin{aligned}
&\psi^{(1)} - y'\phi^{(1)} = \psi^{(2)} - y'\phi^{(2)} = \cdots = \psi^{(m-1)} - y'\phi^{(m-1)} = 0,\\
&\psi^{(m)} - y'\phi^{(m)} + Y^{(m+1)}(\psi_{y'} - y'\phi_{y'} - \phi) = 0.
\end{aligned} \tag{40}$$

THEOREM 8. *The order of contact m of the line of force with the line of E_0 is equal to or greater than one less the order of contact n of the rest trajectory. That is, $m \geqq n-1$.*

Let us assume that $m<n$. Upon substituting the rest trajectory (36) into the expression $\psi - y'\phi$, we find that the power series expansion of this is (since $m<n$)

$$\psi - y'\phi = -\frac{[(\psi_{y'})_0 - \phi_0]C_{m+1}}{m!}x^m + \cdots. \tag{41}$$

Placing this and also the series expansion (36) of the rest trajectory into the differential equation (5), we find that the power of the term of lowest degree is $m+n-2$ (since the condition $m<n$ demonstrates that the term of degree $2n-2$ is higher). Hence upon placing the coefficient of x^{m+n-2} equal to zero, we find

$$(42)\qquad -\frac{[(\psi_{y'})_0-\phi_0]c_{n+1}C_{m+1}(n-1-m)}{m!(n-1)!}=0.$$

Since the angular rate $\lambda\neq 1$, it follows from this equation that if $m<n$, then $m=n-1$. Hence in all possibilities $m\geqq n-1$. Therefore the order of contact of the line of force with the common tangent line can never be less than the order of contact of the rest trajectory decreased by one.

Now let ρ be the ratio of instantaneous departures of the rest trajectory and the line of force from their common tangent line. For the case $m=n-1$, obviously $\rho=0$. Consider next the case where $m=n$. Of course $c_{n+1}\neq 0$ and $C_{n+1}\neq 0$ for $m=n$, but $c_{n+1}\neq 0$ and $C_{n+1}=0$ for $m>n$. The power series expansion of $\psi-y'\phi$ when the rest trajectory is placed therein, is

$$(43)\qquad \psi-y'\phi=\frac{[(\psi_{y'})_0-\phi_0][c_{n+1}-C_{n+1}]}{n!}x^n+\cdots.$$

Placing this and the series expansion (36) of the rest trajectory into our differential equation (5), and putting the coefficient of the term of lowest degree $2n-2$ equal to zero, we discover

$$(44)\qquad \frac{c_{n+1}}{n!(n-1)!}\left[C_{n+1}\{(\psi_{y'})_0-\phi_0\}-c_{n+1}\{(\psi_{y'})_0-(2n+1)\phi_0\}\right]=0.$$

The ratio ρ of departures of the rest trajectory and the line of force from their common tangent line is defined by the ratio $\rho=c_{n+1}/C_{n+1}$. Therefore we derive the following result.

THEOREM 9. *The ratio ρ of the departures from the line of the lineal element E_0 of the rest trajectory and the line of force is given by the formula*

$$(45)\qquad \rho=(1-\lambda)/(2n+1-\lambda)\cdot$$

By Theorem 8, we note that $\rho=0$ if and only if $m=n-1$. By this formula $\rho=\infty$, that is $m>n$ if and only if the angular rate $\lambda=2n+1$. For any other case the orders of contact are the same, $m=n$, and ρ is a finite nonzero number.

For an ordinary field of force, the angular rate $\lambda=0$ so that $\rho=1/(2n+1)$. In the next section, we shall discuss all generalized fields of force for which the ratio $\rho=1/(2n+1)$.

If $\lambda\neq 2n+1$, then the order of contact m with the common tangent line of the line of force is n, or $n-1$, where n is the order of contact of the rest trajectory. However, if $\lambda=2n+1$, then the line of force may have higher order contact with the common tangent line. *This is a new phenomenon.* It can never arise in ordinary positional fields of force. This new situation is that the line of force can have the same order of contact with the common tangent line as that of the rest trajectory, or else one less, or higher.

10. **Generalized fields of force for which the ratio of the departures** $\rho=1/(2n+1)$. By Theorem 9, we see that if $\rho=1/(2n+1)$, then the angular rate $\lambda=0$ at the lineal element E_0.

THEOREM 10. *All generalized fields of force for which the ratio of the departures* $\rho=1/(2n+1)$ *must be such that the eliminant with respect to* y' *of the two equations*

$$\psi - y'\phi, \qquad \psi_{y'} - y'\phi_{y'} = 0, \tag{46}$$

is identically zero.

This theorem means geometrically that the force vector F_0 of the special lineal element E_0 (whose direction coincides with that of F_0) is tangent to the terminal curve C of the point O at the end point of F_0.

11. **Generalized velocity systems.** For our generalized fields of force, we define systems of curves which are generalizations of velocity systems for ordinary positional fields of force. (For the positional case, see the Princeton Colloquium, pp. 42–44.)

Consider a particle of unit mass in any generalized positional field of force with rectangular components ϕ and ψ. The equations of motion are then given by (1). If the initial position and the initial velocity are given, the motion is determined. If only the initial position and the direction of the motion are given, the radius of curvature r will depend for its value on the initial speed. Hence, in addition to the usual formula for the speed v, there must be a formula expressing v^2 in terms of x, y, y', r. This is furnished by the first of equations (3). This equation may be written in the form

$$v^2 = (\psi - y'\phi)(1 + y'^2)/y'' \cdot \tag{47}$$

In the actual trajectory v varies from point to point. If now we replace v^2 in this result by some constant, say $1/c$, the resulting equation may be written in the form

$$y'' = c(\psi - y'\phi)(1 + y'^2). \tag{48}$$

The curves satisfying this differential equation, they are not in general trajectories, we define as *generalized velocity systems*. For any generalized field of force, a curve is a velocity curve corresponding to the speed v_0, provided a particle starting from any lineal element of the curve with that speed describes a trajectory osculating the curve. In a given field of force there are ∞^3

trajectories and ∞^3 velocity curves. If c is given, we have ∞^2 velocity curves. For unit velocity, we have the following result.

THEOREM 11. *Any system of ∞^2 curves may be regarded as the totality of velocity curves corresponding to unit velocity in $\infty^{f(3)}$ generalized fields of force.*

Upon eliminating the constant c, we obtain the complete system of ∞^3 velocity curves for a given generalized field of force. The differential equation of all velocity curves is

$$(49)\qquad \begin{aligned}(\psi - y'\phi)y''' = {} & [\psi_x + y'(\psi_y - \phi_x) - y'^2\phi_y]y'' \\ & + [\psi_{y'} - y'\phi_{y'} - 3\phi + 2(\phi + y'\psi)/(1 + y'^2)]y''^2.\end{aligned}$$

Therefore a complete velocity system is of the type (G), thus possessing the Property I.

Observe that the expression $(\psi - y'\phi)(1+y'^2)$ of equation (48) can represent any function of (x, y, y'). This result shows that any complete velocity system is identical with curvature trajectories(4). For an arbitrary family of ∞^2 curves, a curve is a *curvature trajectory* if at any lineal element E of this curve, its curvature is c times the curvature of the curve of the family which passes through E. For a given family of ∞^2 curves, there are ∞^3 curvature trajectories. These possess an additional differential property in addition to the Property I.

THEOREM 12. *Any system of ∞^3 curvature trajectories may represent the velocity curves of $\infty^{1+f(3)}$ generalized fields of force.*

By equations (6), it is seen that the velocity systems are the curvature trajectories

$$(50)\qquad y''' = (\alpha_x + y'\alpha_y)y'' + \alpha_{y'}y''^2.$$

By this, it is seen that for a given system of ∞^3 curvature trajectories, the function α is completely determined except for an additive constant. Since β is entirely arbitrary, we therefore discover the validity of our theorem.

12. **The generalized systems S_k defined by $P = kN$.** In a generalized field of force, we may define systems of curves S_k which are generalizations of the corresponding systems S_k for the ordinary positional case (see the Princeton Colloquium, pp. 91–94).

If an arbitrary curve is drawn in a generalized field of force, and the particle of unit mass is started along it from one of its points with a given speed, the constrained motion along the given curve is determined. The acceleration along the curve is given by T, the tangential component of the force vector. So the speed at any point is determined by

$$(51)\qquad v^2 = 2\int T\,ds.$$

The pressure P (of course normal to the curve, since the curve is considered smooth) is given by the formula

$$P = v^2/r - N. \tag{52}$$

If we increase the initial speed, the effect is to increase v^2 by a constant c; and hence P changes by the addition of a term c/r.

If the given curve is a trajectory, the initial speed may be so chosen that the pressure vanishes throughout the motion; that is, trajectories may be defined as curves of no constraint. Of course if a different initial speed is used, P will be of the form c/r; but, as regards the curves, they are completely characterized by $P=0$.

The general problem that we wish to consider is to find curves so that the pressure P shall be proportional to the normal component N of the force vector. So $P=kN$. To a given value of k there correspond ∞^3 such curves. The system so obtained will be denoted by S_k.

We note that $k=0$ gives the system S_0 of trajectories and $k=\infty$ gives the system S_∞ of velocity curves.

To derive the differential equation of the system S_k, we must eliminate the speed v from the equations

$$v^2 = (k+1)rN, \qquad vv_s = T. \tag{53}$$

The result is

$$Nr_s = nT - rN_s, \tag{54}$$

where $n=2/(k+1)$.

Expanding this, we discover as the differential equation of the system S_k

$$\begin{aligned}(\psi - y'\phi)y''' &= [\psi_x + y'(\psi_y - \phi_x) - y'^2\phi_y]y'' \\ &\quad + [\psi_{y'} - y'\phi_{y'} - 3\phi - (n-2)(\phi + y'\psi)/(1+y'^2)]y''^2.\end{aligned} \tag{55}$$

We note that $n=2$ gives the generalized dynamical trajectories, and $n=0$ gives the velocity curves.

THEOREM 13. *Any system S_k is represented by a differential equation of the type* (G) *characterized by the geometric Property* I. *Conversely, any system of ∞^3 curves with the Property* I *can represent the system S_k with $k\neq\infty$ of $\infty^{f(2)}$ generalized fields of force.*

In terms of the functions α and β defined by equations (6), our system S_k is represented by the differential equation of the type (G)

$$y''' = (\alpha_x + y'\alpha_y)y'' + (\alpha_{y'} + (n/2)\beta)y''^2. \tag{56}$$

Thence if $n\neq 0$, the functions α and β may be determined up to an arbitrary function of two variables.

We note that this theorem is not valid for velocity curves $k=\infty$ or $n=0$. In addition to Property I, the type (G) must possess the additional differential conditions of second order

$$(57)\qquad \begin{aligned} G_{yy'} &= 2H_y + H_{xy'} + y'H_{yy'}, \\ G_{xy'} + y'G_{yy'} - G_y &= H_{xx} + 2y'H_{xy} + y'^2H_{yy}. \end{aligned}$$

If these conditions are satisfied, then our differential equation of the type (G) represents curvature trajectories. These may therefore represent the velocity curves of $\infty^{1+f(3)}$ generalized fields of force.

13. **Curves of constant pressure.** We now define curves of constant pressure in a generalized field of force analogous to those in an ordinary positional field of force. (See the Princeton Colloquium, pp. 97–98.) Consider the curves corresponding to $P=c$, where c denotes any constant. The curves obtained may be termed curves of constant pressure. Only along such a curve is a constrained motion of a particle possible such that the pressure against the curve remains constant.

For a given value of c, a system of ∞^3 curves is obtained, whose intrinsic equation, found by differentiating the relation

$$(58)\qquad P = v^2/r - N = c,$$

is

$$(59)\qquad (c + N)r_s = 2T - rN_s.$$

By this, we see that this system is given by a differential equation of the type (G) and therefore possesses the geometric Property I.

THEOREM 14. *For a given value of c, there are ∞^3 pressure curves $P=c$ whose differential equation is of the type* (G) *characterized by the geometric Property* I. *Conversely, any system of ∞^3 curves with the Property* I *may represent the pressure curves for a preassigned c of $\infty^{f(2)}$ generalized fields of force.*

This result may be deduced from the differential equation of the pressure curves $P=c$, which may be written by means of equations (6) in the form

$$(60)\qquad \begin{aligned} [1 + c(1 + y'^2)^{3/2}]y''' = {} & [\alpha_x + y'\alpha_y]y'' \\ & + [\beta + \alpha_{y'} + 3cy'(1 + y'^2)^{1/2}e^{-\alpha}]y''^2. \end{aligned}$$

Thence for a preassigned c, α is determined up to an arbitrary function of two variables and β is thereby completely known.

If all systems S_k are combined, we obtain ∞^4 curves. Similarly combining all pressure curves, we find ∞^4 curves. Each of these *two distinct* quadruply-infinite families of curves is given by a differential equation of the form

(61) $$y^{\mathrm{IV}} = Ay'''^2 + By''' + C,$$

where A, B, C are functions of the curvature element (x, y, y', y''). (See the Princeton Colloquium, pp. 95, 98, 113–115.)

The system (61) may be characterized geometrically as follows. If for each of the ∞^1 curves of this family which pass through a given curvature element (x, y, y', y''), we construct the osculating conic (five-point contact), the locus of the centers of these conics is a conic passing through the given point in the given direction.

Such quadruply-infinite systems of curves (61) were first encountered in connection with the systems of extremals in a calculus of variation problem(9). These differential equations also arise in connection with the study of special types of fields of force which depend upon the position of the point and the time only, and in many other connections(10).

In conclusion, we note that all our results are valid under the total projective group consisting of collineations and correlations. The only contact transformations of the plane which preserve differential equations of the type (G) are the projective ones.

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(9) Kasner, *Systems of extremals in the calculus of variations*, Bull. Amer. Math. Soc. vol. 13 (1907) p. 290.

(10) See Kasner, Revista de Matematicas vol. 3 (1942) pp. 7–12.

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